# Classifying nuclear C*-algebras

(a brief look)

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## I. Prologue.

- consider $M_2 \subset M_4 \subset M_8 \subset \cdots \subset \bigcup_{n\geq 1} M_{2^n}$, $\bigcup_{n\geq 1} M_{3^n}$
  $x \mapsto \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$
- M-vN '43: $\overline{\bigcup_{n\geq 1} M_{2^n}}^{WOT} \cong \overline{\bigcup_{n\geq 1} M_{3^n}}^{WOT}$
- Glimm '60: $\underbrace{\overline{\bigcup M_{2^n}}^{\|\cdot\|}}_{M_{2^\infty}} \not\cong \underbrace{\overline{\bigcup M_{3^n}}^{\|\cdot\|}}_{M_{3^\infty}}$ "UHF alg's"

How to tell them apart? K-theory!

## II. Reminder: $K_0(A)$

- $K_0(A) = \{ [p]_0 - [q]_0 : p, q \in \mathrm{Proj}_\infty(A) \}$ unital
  $[p]_0 = [q]_0$ iff $[\, \underbrace{p \oplus r}_{vv^*} \sim \underbrace{q \oplus r}_{v^*v} \,]$ cancellation: don't need r
- $K_0(-)$ is "nice", e.g. $K_0(\varinjlim A_n) \cong \varinjlim K_0(A_n)$

Ex. $K_0(\overbrace{M_{2^\infty}}^{\text{CAR algebra}}) = \{ \frac{m}{2^n} : m, n \in \mathbb{Z} \}$, $[1]_0 = 1$

## III Elliott's classif. of AF alg's

- AF: $\overline{\bigcup \text{fin. dim'l}}^{\|\cdot\|}$ ⊕ matrix alg's

**Theorem** '76: sep., unital $A, B$: AF alg's. Then

$$A \cong B \iff (K_0(A), \underbrace{K_0(A)_+}_{\{[p]_0 \,:\, p \in \mathrm{Proj}_\infty(A)\}}, [1_A]_0) \cong \overbrace{(K_0(B), K_0(B)_+, [1_B]_0)}^{\mathrm{inv}(B)}$$

General idea: study $*$-hom's "$\Longleftarrow$"

1) Existence: from $\mathrm{inv}\, A \xrightarrow{\varphi} \mathrm{inv}\, B$, produce $*$-hom $A \xrightarrow{\Phi} B$
2) Uniqueness: show that $\Phi$ is essentially unique, up to some $\approx$
3) Argument: $\mathrm{inv}\, A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{inv}\, B$ inverses $\rightsquigarrow$ $A \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} B$ $*$-hom's; inverses up to $\approx$: good enough to show $A \cong B$.

## IV Plausibility of existence.

- imagine $\varphi : K_0(M_{2^\infty}) \longrightarrow K_0(B)$
- reduce $\varphi : K_0(M_{2^N}) \longrightarrow K_0(F)$ fin. dim'l
  pretend $N = 1$.

(i) $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Can find proj'ns $f_{11}, f_{22} \in F$ s.t. $\varphi([e_{ii}]_0) = [f_{ii}]_0$ & $f_{11} \perp f_{22}$. (Cancellation!)

(ii) $e_{11} \sim e_{22} \Rightarrow [e_{11}]_0 = [e_{22}]_0 \Rightarrow [f_{11}]_0 = [f_{22}]_0 \Rightarrow \underbrace{f_{11}}_{vv^*} \sim \underbrace{f_{22}}_{v^*v}$

Let $f_{12} := v$, $f_{21} = v^*$.

"Define" $\Phi$ by $e_{ij} \mapsto f_{ij}$. QED ish.

## V. High-tech application of a modern existence thm.

discrete

$G$ is amenable

$\Updownarrow$

$C^*_r(G) \hookrightarrow \mathcal{Q}$

(∃ an inj. $*$-hom. that preserves the trace)

lim of $M_2 \subset M_{3!} \subset M_{4!} \subset M_{5!} \cdots$
$K_0(\mathcal{Q}) = \mathbb{Q}$

# VI. Where do we stand ... and how did we get here? (a biased take)

- generalizations? inspiration from vNa side (Connes '76)
- Elliott: classifies large class of AT alg's — lim's of $\oplus M_{n_k}(C(\mathbb{T}))$, conjectures classification holds very generally (simple, nuclear, ...)
- positive results for more general types of lim's; includes alg's not obviously of that form — e.g. $A_\theta$

- also challenges: Rørdam, Toms produced counterexamples; not possible to fix by modifying invariant
- Toms-Winter regularity: study properties that make a $C^*$-alg. "well behaved"
- analogy with vNa factors, where natural notions of amenability are equiv.

injective — hyperfinite — McDuff — tracial comparison of proj'ns

## The Toms-Winter ~~conjecture:~~ essentially a thm.

- A: unital, sep., simple, nuclear, $\neq M_n(\mathbb{C})$

TFAE:

(i) A has finite $\boxed{\text{nuclear dimension}}$

(ii) A absorbs the Jiang-Su alg. $\mathcal{Z}$ tensorially: $\boxed{A \otimes \mathcal{Z} \cong A}$

(iii) A has $\boxed{\text{strict comparison}}$ of positive elements

- NC analog of covering dim: $\dim_{nuc} C(X) = \dim X$
- connections to dim theories for groups, dyn. systems ...

- analog of McDuff: $M \bar{\otimes} \mathcal{R} \cong M$
- $\infty$-dim'l analog of $\mathbb{C}$; K-theory can't tell them apart

- for proj'ns: tracial states can determine if $p \leq q$; analog for positive elements (with a different $\precsim$ than $\leq$).

## Classification Theorem:

$\pi(\underbrace{C^*(\text{nilpotent grp})})$

- $A, B$: unital, sep., simple, nuclear, $\mathcal{Z}$-stable, satisfying UCT
- $A \cong B$ iff $(K_0(A), [1_A]_0, K_1(A), T(A), \rho_A) \cong$

$$\cong (K_0(B), [1_B]_0, K_1(B), T(B), \rho_B)$$

**One strategy** (C - Gabe - Schafhauser - Tikuisis - White)

"Lift" classification theorems in vNa setting in order to prove existence & uniqueness theorems in $C^*$-setting.